Lines and circles joining components of a link

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Abstract We estimate from below the number of lines meeting each of given 4 disjoint smooth closed curves in a given cyclic order in the real projective 3-space and in a given linear order in $\mathbb{R}^3$. Similarly, we estimate the number of circles meeting in a given cyclic order given 6 disjoint smooth closed curves in Euclidean 3-space. The estimations are formulated in terms of linking numbers of the curves and obtained by orienting of the corresponding configuration spaces and evaluating of their signatures. This involves a study of a surface swept by lines meeting 3 given disjoint smooth closed curves and a surface swept in the 3-space by circles meeting 5 given disjoint smooth closed curves. Higher dimensional generalizations of these results are outlined.

AMS Classification 57M25

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1 Introduction

Consider several disjoint smooth closed curves in the 3-space. If they are not linked to each other, one can pull them apart by an isotopy in such a way that any line joining two of them would not meet any other one. Of course, any three of the curves can be still connected by a circle, since a circle can be drawn through any three non-colinear points, but one can eliminate circles connecting more than three of the curves.

However, as it is proved below, if the curves are linked well enough to each other, any 4 of them can be connected by a line, and any 6, by a circle. Moreover, the numbers of such lines and circles are estimated from below in terms of pairwise linking numbers of the curves.

1.1 In Projective 3–Space

Linking Numbers. Recall that linking number $\operatorname{lk}(A,B)$ is defined for any pair of disjoint oriented closed curves A, B in an oriented closed 3–manifold

M realizing homology classes $[A], [B] \in H_1(M)$ of finite order. It is defined as $\frac{1}{n} B \circ C$, where C is a smooth chain transversal to B with boundary $\partial C = nA$, and $\circ$ denotes the intersection number. Modulo one, linking number depends only on the homology classes of the curves. This defines a well-known symmetric bilinear form $\operatorname{Tors} H_1(M) \times \operatorname{Tors} H_1(M) \to \mathbb{Q}/\mathbb{Z}$, which is a part of Poincaré duality.

In particular, linking numbers of curves in the three–dimensional real projective space $\mathbb{R}P^3$ are integers or half-integers. Half-integer appears iff both curves are not zero-homologous in $\mathbb{R}P^3$.

Theorem 1 (On Projective Lines Meeting 4 Curves, Pedestrian Version) *Let C_1, C_2, C_3, C_4 be disjoint oriented smooth closed curves in $\mathbb{R}P^3$ in general position. Each real projective line L intersecting the curves in this cyclic order can be equipped with a weight $\omega(L) = \pm 1$ such that*

$$\sum_L \omega(L) = 2\left(\operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4) - \operatorname{lk}(C_2, C_3)\operatorname{lk}(C_4, C_1)\right),$$

where the sum runs over the set of all projective lines L intersecting curves C_1, C_2, C_3, C_4 in this cyclic order.

General Position Conditions and Their Roles. In fact there are three general position conditions on C_1, C_2, C_3, C_4 which are important:

(1) The number of lines intersecting all of them is finite.

(2) There is no line which meets the union $\cup_{i=1}^4 C_i$ in five points.

(3) For each line L meeting each of C_i, the lines L_i tangent to C_i at point $L \cap C_i$ are pairwise non-coplanar.

(4) For each line L meeting each of C_i, there is no surface of degree 2 containing L and tangent to C_i at p_i for each $i = 1, 2, 3, 4$.

The first condition is crucial for making the sum of weights finite. The second, third and fourth conditions are not that necessary, but without accepting them we would not be able to claim that the weight of each line is ± 1.

Set of Projective Lines Visiting Sets. Let A, B, C and D be subsets of a real projective space. Denote by $\mathcal{P}(A, B, C, D)$ the set of projective lines which pass through points $a \in A$, $b \in B$, $c \in C$, $d \in D$ in this cyclic order.

Recall that a real projective line is homeomorphic to circle. A set of four points on a circle inherits two *cyclic* orders, each of which corresponds to an orientation of the circle. The set $\mathcal{P}(A, B, C, D)$ consists of those projective line for which

one of these two cyclic orders induced from the line on $\{a, b, c, d\}$ coincides with the prescribed cyclic order.

Interpretation of Theorem 1 From Viewpoint of Differential Topology. The general position assumptions imply that $\mathcal{P}(C_1, C_2, C_3, C_4)$ is a 0–dimensional manifold. Theorem 1 claims that this manifold has a natural orientation. Recall that an orientation of 0–manifold is nothing but a function on it with values ± 1. The sum of all the values of an orientation ω of a compact 0–manifold M is the signature $\operatorname{sign}(M, \omega)$ of M. Hence Theorem 1 admits the following reformulation.

Theorem 1 Reformulated *For any disjoint oriented smooth closed curves $C_1, C_2, C_3, C_4 \subset \mathbb{R}P^3$ in general position, the set $\mathcal{P} = \mathcal{P}(C_1, C_2, C_3, C_4)$ is a 0–manifold, which admits a natural orientation ω with signature*

$$\operatorname{sign}(\mathcal{P}, \omega) = 2\left(\operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4) - \operatorname{lk}(C_2, C_3)\operatorname{lk}(C_4, C_1)\right).$$

The Role of Order. In Theorem 1 curves C_i are not assumed to be connected. $C_1 \cup C_2 \cup C_3 \cup C_4$ can be considered as an oriented link in $\mathbb{R}P^3$ with components partitioned into four disjoint groups. A link with such partition is called a colored link. The colors (i.e., the groups of components) are assumed to be linearly ordered, and the order works in several ways.

Obviously, only those projective lines are considered, which meet components of the link in the cyclic order of their colors. It does not make sense to speak about the linear order in which four points are positioned on a circle.

However, the linear order of the curves cannot be replaced in Theorem 1 by a cyclic order. The weights which the projective lines are equipped with depend on the linear order. Even the expression for the sum of the weights changes sign under cyclic permutation $(1, 2, 3, 4)$ of curves C_i.

Low Bounds for the Number of Lines Meeting Four Curves. Theorem 1 gives low bounds for the number of lines intersecting four given closed oriented curves in $\mathbb{R}P^3$:

Corollary 1 of Theorem 1 *Under hypothesis of Theorem 1, the number of real projective lines intersecting curves C_1, C_2, C_3, C_4 in this cyclic order (i. e., the number of points in $\mathcal{P}(C_1, C_2, C_3, C_4)$) is at least*

$$2\left|\operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4) - \operatorname{lk}(C_2, C_3)\operatorname{lk}(C_4, C_1)\right|.$$

Summing up estimates provided by this for all cyclic orders, we get the following total estimate:

Corollary 2 of Theorem 1 *Under hypothesis of Theorem 1, the total number of real projective lines intersecting each of C_i is at least*

$$4 \max_{\substack{(i,j,k,l) \text{ is a} \\ \text{permutation} \\ \text{of } (1,2,3,4)}} |\operatorname{lk}(C_i, C_j)\operatorname{lk}(C_k, C_l) - \operatorname{lk}(C_j, C_k)\operatorname{lk}(C_l, C_i)|$$

The Role of Orientations of the Curves. Orientations of curves C_i are needed in Theorem 1 for making linking numbers $\operatorname{lk}(C_i, C_j)$ defined. By changing the orientations, one may hope to improve the low bound in some cases. However, if each of C_i is connected, the low bound does not depend on the orientations. Indeed, reversing the orientation of any single curve changes the signs of both summands in the expression $\operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4) - \operatorname{lk}(C_2, C_3)\operatorname{lk}(C_4, C_1)$.

Examples: Links Made of 4 Lines. There are three isotopy classes of links in $\mathbb{R}P^3$ made of four disjoint projective lines, see [5]. Their representatives are shown in Figure 1.

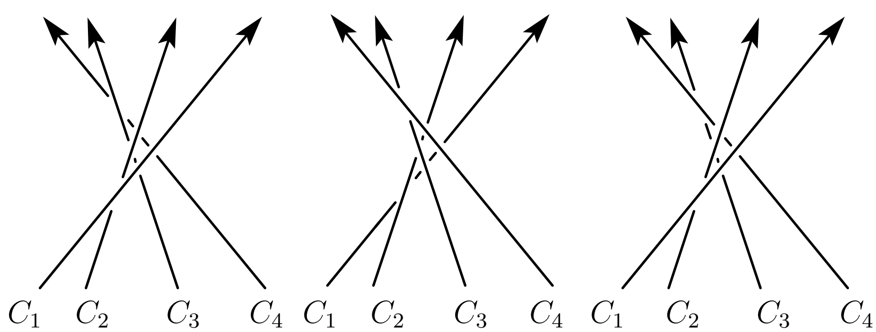

Figure 1: Links consisting of four disjoint lines in $\mathbb{R}P^3$, which represent all three isotopy classes of such non-oriented links. The links shown on the left hand side and in the middle are mirror images of each other. The link on the right hand side is amphicheiral. Orientations are shown to make $\operatorname{lk}(C_i, C_j)$ defined.

In the link shown on the left hand side, $\operatorname{lk}(C_i, C_j) = +\frac{1}{2}$ for any i and j, in the link shown in the middle $\operatorname{lk}(C_i, C_j) = -\frac{1}{2}$ for any i and j. Hence, in both cases Corollary 2 of Theorem 1 gives the trivial low bound for the number of lines meeting each of C_i. *This low bound is exact: in both cases the isotopy class contains a link for which there is no line meeting all for components.*

Indeed, both configuration of lines can be placed on a hyperboloid, as a collection of 4 lines belonging to a family of pairwise disjoint lines which cover the whole hyperboloid. See Figure 2. For this representative of the isotopy class there are infinitely many lines meeting all 4 components. Namely, each line of

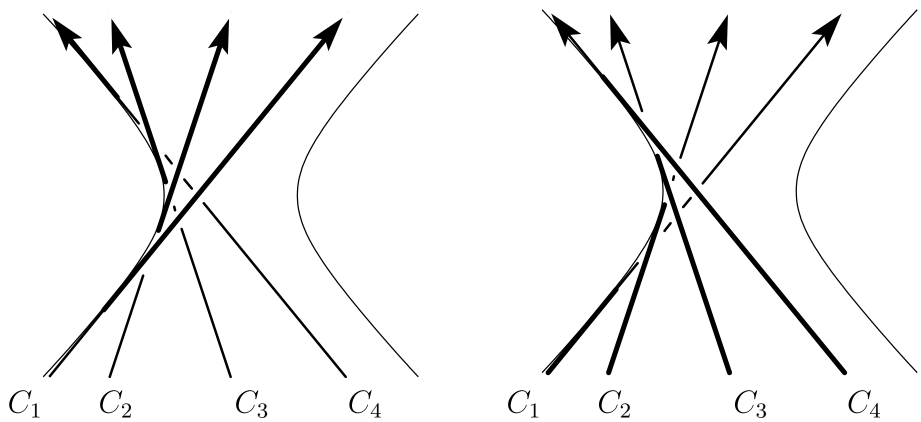

Figure 2: Two links made of 4 lines on a hyperboloid.

the other family of lines covering the hyperboloid meets each line of the first family of lines. However, one can move one component of the link into one of the domains bounded by the hyperboloid. See Figure 3. Any line which meets each of the other three lines must lie on the hyperboloid, because it meets it in at least three points. Hence it has no chance to meet the line which was taken off the hyperboloid.

In the rightmost link of Figure 1, $\mathrm{lk}(C_3, C_4) = -\frac{1}{2}$ and $\mathrm{lk}(C_i, C_j) = +\frac{1}{2}$ for any other values of i, j. Hence, Corollary 2 of Theorem 1 claims that there are at least $4|\frac{1}{2}(-\frac{1}{2}) - \frac{1}{2}\frac{1}{2}| = 4| - 2\frac{1}{4}| = 2$ lines meeting lines C_1, C_2, C_3 and C_4. According to Corollary 1, at least one of them meet the lines in cyclic order C_1, C_2, C_3, C_4, and at least one, in cyclic order C_1, C_2, C_4, C_3.

These bounds are also exact. To see this, consider the hyperboloid H swept by lines meeting C_1, C_2 and C_3. See Figure 4. The fourth line, C_4, intersects the hyperboloid in two points. The intersection points are in two connected components of $H \smallsetminus (C_1 \cup C_2 \cup C_3)$ which are adjacent to C_3. This is seen in Figure 4, and this is the only possibility. Indeed, if C_4 did not meet H at all, by an isotopy inverse to the one shown in Figure 3 one could put entirely on H, but then the linking numbers would be as in one of two other cases shown in Figure 2. If the intersection points were in the same connected component of $H \smallsetminus (C_1 \cup C_2 \cup C_3)$, moving the points towards each other one would construct an isotopy of the link with C_4 tangent to H. This would give again one of two other links. Finally, one of the points of $C_4 \cap H$ cannot be in the component of $H \smallsetminus (C_1 \cup C_2 \cup C_3)$ bounded by C_1 and C_2, because then the equality $\mathrm{lk}(C_1, C_4) = \mathrm{lk}(C_2, C_4)$ would not hold. In fact, these arguments prove the

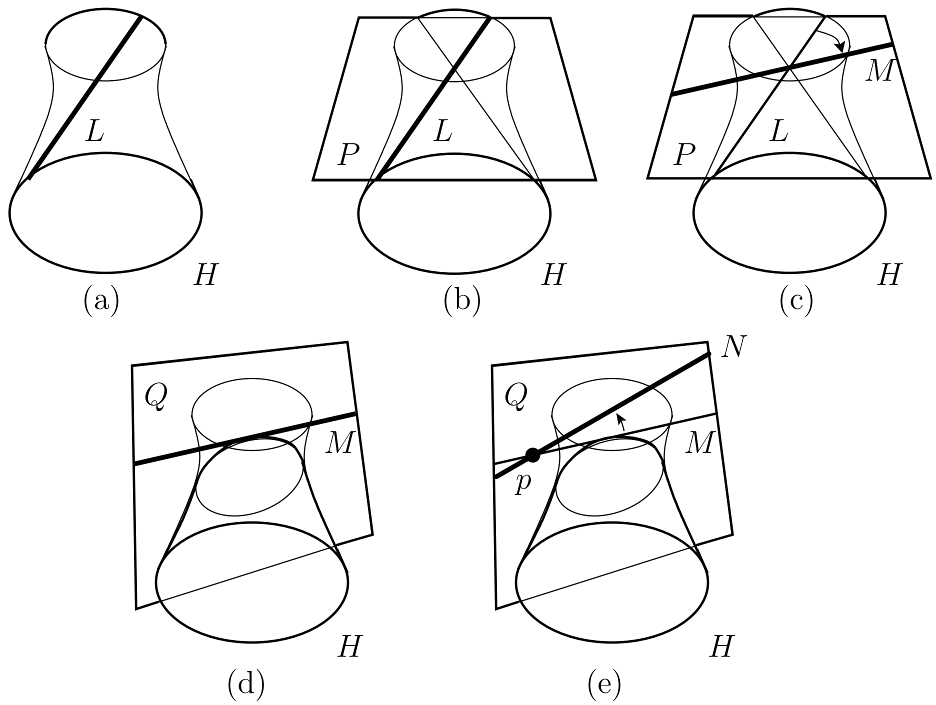

Figure 3: Moving line L off hyperboloid $H \supset L$. In the projective space such a move cannot be made in a single plane. Here it is presented as a composition of two rotations in different planes. Draw a plane P containing L, see (b). Plane P is tangent to H at some point. Rotate line L in plane P around the point, see (c). The result M is a line tangent to H. Draw another plane Q containing M and intersecting H in a non-singular conic, see (d). Rotate M in plane Q around a point $p \in M \smallsetminus H$.

isotopy classification of links in $\mathbb{R}P^3$ consisting of four lines. For details see [5].

The lines meeting C_1, C_2, C_3, and C_4 lie on H and pass through the intersection points of C_4 and H. The line passing through the point which lies on H between C_2 and C_3 meets the lines in cyclic order C_1, C_2, C_4, C_3. The line passing through the point which lies on H between C_3 and C_1 meets the lines in cyclic order C_1, C_2, C_3, C_4.

Absence of Upper Bounds. Under hypothesis of Theorem 1, there is no upper bound for the number of real projective lines intersecting curves C_1, C_2, C_3 and C_4 in terms of linking numbers or any other invariants, which do not change under isotopy of the curves.

Indeed, one can draw a line L connecting a point on C_1 with a point on C_2, then move C_3 and C_4 by an isotopy to make them meeting L, so that L would

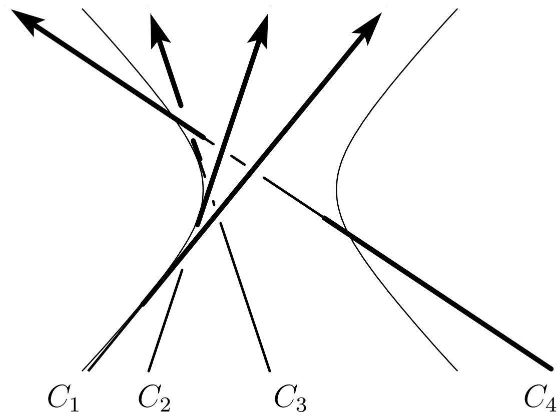

Figure 4: Amphicheiral link of 4 lines and the hyperboloid swept by lines meeting three of its components.

meet C_1, C_2 and C_3 in a prescribed cyclic order. Then one can make short pieces of C_1, C_2 and C_3 near their intersection points with L line segments. The set of lines meeting three given lines covers a quadric surface. A strip of this surface is formed by lines meeting the rectilinear pieces of C_1, C_2 and C_3. By a small isotopy of C_4 in a neighborhood of $C_4 \cap L$ one can force C_4 to meet this strip arbitrary number of times near the point $C_4 \cap L$. Each of the intersection points is on one of the lines which form the strip. These lines meet C_1, C_2, C_3 and C_4 in the prescribed cyclic order.

Upper bounds exist in the case when the curves are defined by algebraic equations and admit complexification. Then the complexification provides terms in which upper bounds can be formulated. We will return to this, but we need, first, to consider the corresponding complex problems.

Similar Problem from Complex Algebraic Geometry. What happens to the problems discussed above, if we pass from smooth curves in $\mathbb{R}P^3$ to complex curves in the complex projective 3–space $\mathbb{C}P^3$? For generic C_1, C_2, C_3 and C_4 there is finitely many complex projective lines meeting each of C_i. Points on a complex projective line do not have any distinguished cyclic order. Hence, it does not make sense to consider lines which meet the curves in whatever cyclic order. However one may wonder about the total number of all complex projective lines intersecting each of four given disjoint generic algebraic curves C_1, C_2, C_3, C_4. This is a classical problem of complex enumerative algebraic geometry, and its solution is well-known.

As this is the complex algebraic geometry, estimates are replaced by an exact

value. It is equal to $2d_1d_2d_3d_4$, where d_i is the order of C_i, that is the intersection number of C_i with a generic hyperplane. (If the curves are not in general position, there may be infinitely many lines, or the number of lines counted with positive multiplicities is still $2d_1d_2d_3d_4$.)

To prove this, remark, first, that the number of lines meeting each of C_1, C_2, C_3, and C_4 equals the number of intersection points of C_4 and the surface swept by lines meeting each of curves C_1, C_2 and C_3. By the Bezout theorem this number is the product of d_4 by the degree of the surface. Therefore it depends linearly on d_4. By symmetry, it depends linearly on each of d_i. Thus, this is $kd_1d_2d_3d_4$. In the case, when each of the curves C_i is a line, the surface swept by lines meeting each of C_1, C_2, C_3 is a hyperboloid, hence $k = 2$. □

In Real Algebraic Geometry. If under assumptions of Theorem 1 curves C_1, C_2, C_3 and C_4 are algebraic, by taking complexification of all the varieties involved we obtain the complex algebraic situation considered above. Some of $2d_1d_2d_3d_4$ complex projective lines meeting the complexifications of C_1, C_2, C_3 and C_4 may be imaginary (not real), but the complexification of any real line meeting C_1, C_2, C_3 and C_4 is among these $2d_1d_2d_3d_4$ complex projective lines. Thus, in contrast to purely topological situation discussed above, in real algebraic situation there is an upper bound: the total number of real lines meeting real algebraic curves C_1, C_2, C_3 and C_4 in $\mathbb{R}P^3$ is at most $2d_1d_2d_3d_4$, where d_i is the order of C_i.

1.2 In Affine 3-Space

Results on links in $\mathbb{R}P^3$ are applied to links in $\mathbb{R}^3$, just because $\mathbb{R}^3$ is embedded in $\mathbb{R}P^3$. Besides, there appear new opportunities related to natural linear orders of points on a line.

Let A, B, C and D be subsets of an affine space. Denote by $\mathcal{A}(A, B, C, D)$ the set of affine lines which pass through points $a \in A$, $b \in B$, $c \in C$, $d \in D$ in this linear order.

Notice that sets A, B, C, D are not assumed to be pairwise distinct.

Theorem 2 (On Affine Lines Meeting 4 Curves) *For any disjoint oriented smooth closed curves $C_1, C_2, C_3, C_4 \subset \mathbb{R}^3$ in general position, the set $\mathcal{A} = \mathcal{A}(C_1, C_2, C_3, C_4)$ of lines meeting curves C_1, C_2, C_3, C_4 in their order is a 0–manifold, which admits a natural orientation ω with signature*

$$\operatorname{sign}(\mathcal{A}, \omega) = \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_4).$$

As I was informed by Oleg Viro, this theorem was proven using different arguments by Michael Polyak (unpublished).

Almost everything written in the preceding section about Theorem 1 can be repeated here concerning Theorem 2. In particular, the general position conditions are exactly the same.

Notice that in Theorem 2 lines under consideration meet the curves in a fixed *linear* order. There are two linear orders for points on a line corresponding to orientations of the line, and here any of these two orders is meant. The transition from cyclic to linear order is responsible for the difference between expressions for the sum of weights in theorems 1 and 2. Theorem 2 applied to all linear orders corresponding to the same cyclic order implies Theorem 1 for the case when all the curves are contained in an affine part of the projective space.

Low Bounds for the Number of Lines Meeting Four Curves. As the absolute value of signature of a 0–manifold cannot exceed the number of points, Theorem 2 gives low bounds for the number of lines intersecting four given closed oriented curves in $\mathbb{R}^3$:

Corollary 1 of Theorem 2 *Under hypothesis of Theorem 2, the number of lines intersecting curves C_1, C_2, C_3, C_4 in this order is at least*

$$|\operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4)|.$$

Summing up the estimates provided by this for all orders, we get the following total estimate:

Corollary 2 of Theorem 2 *Under hypothesis of Theorem 2, the total number of lines intersecting curves C_1, C_2, C_3, C_4 is at least*

$$4\left(|\operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4)| + |\operatorname{lk}(C_1, C_3)\operatorname{lk}(C_2, C_4)| + |\operatorname{lk}(C_1, C_4)\operatorname{lk}(C_2, C_3)|\right).$$

Connecting Two or Three Curves. The next two theorems are specific for affine case, their projective counter-parts are trivial.

Theorem 3 (On Affine Lines Meeting 2 Curves Intermittingly) *For any disjoint oriented smooth closed curves C_1 and C_2 in $\mathbb{R}^3$ in general position, the set $\mathcal{A} = \mathcal{A}(C_1, C_2, C_1, C_2)$ is a 0–dimensional manifold, which admits an orientation ω with signature*

$$\operatorname{sign}(\mathcal{A}, \omega) = (\operatorname{lk}(C_1, C_2))^2.$$

Theorem 4 (On Affine Lines Meeting 3 Curves Intermittingly) *For any disjoint oriented smooth closed curves C_1, C_2 and C_3 in $\mathbb{R}^3$ in general position, the sets $\mathcal{A}_{1213} = \mathcal{A}(C_1, C_2, C_1, C_3)$ and $\mathcal{A}_{1231} = \mathcal{A}(C_1, C_2, C_3, C_1)$ are 0–manifolds, which admit orientations ω_1 and ω_2, respectively, such that*

$$\operatorname{sign}(\mathcal{A}_{1213}, \omega_1) = \operatorname{sign}(\mathcal{A}_{1231}, \omega_2) = \operatorname{lk}(C_1, C_2)\operatorname{lk}(C_1, C_3).$$

1.3 Connecting by Circles

In theorems which follow the role that is played by projective or affine lines above passes to circles. The space of all circles in $\mathbb{R}^3$ is a manifold of dimension 6. A condition that a circle meets a fixed generic curve specifies a hypersurface in the space of all circles. A transversal intersection of 6 such hypersurfaces is a 0-dimensional space. This is why the number of curves is increased to 6.

Let $A_1, \ldots, A_n$ be subsets of an Euclidean space. Denote by $\mathcal{S}(A_1, \ldots, A_n)$ the set of circles and lines which pass through points $a_1 \in A_1, \ldots, a_n \in A_n$ in this cyclic order.

Theorem 5 (On Circles Meeting 6 Curves) *For any disjoint oriented smooth closed curves C_1, C_2, C_3, C_4, C_5, C_6 in $\mathbb{R}^3$ in general position, the set $\mathcal{S} = \mathcal{S}(C_1, C_2, C_3, C_4, C_5, C_6)$ is a 0–manifold, which admits an orientation ω with signature*

$$\operatorname{sign}(\mathcal{S}, \omega) = \operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_4)\operatorname{lk}(C_5, C_6) - \operatorname{lk}(C_2, C_3)\operatorname{lk}(C_4, C_5)\operatorname{lk}(C_6, C_1).$$

Theorem 6 (Circles Meeting 5 Curves) *For any disjoint oriented smooth closed curves C_1, C_2, C_3, C_4, C_5 in $\mathbb{R}^3$ in general position, sets*

$$\begin{aligned} \mathcal{S}_{121345} &= \mathcal{S}(C_1, C_2, C_1, C_3, C_4, C_5) \\ \mathcal{S}_{123145} &= \mathcal{S}(C_1, C_2, C_3, C_1, C_4, C_5) \end{aligned}$$

are 0–manifolds, which admit orientations ω_1 and ω_2, respectively, such that

$$\begin{aligned} &\operatorname{sign}(\mathcal{S}_{121345}, \omega_1) = \\ &\qquad \operatorname{lk}(C_1, C_2)\operatorname{lk}(C_1, C_3)\operatorname{lk}(C_4, C_5) - \operatorname{lk}(C_2, C_1)\operatorname{lk}(C_3, C_4)\operatorname{lk}(C_5, C_1) \\ &\operatorname{sign}(\mathcal{S}_{123145}, \omega_2) = \\ &\qquad \operatorname{lk}(C_1, C_2)\operatorname{lk}(C_3, C_1)\operatorname{lk}(C_4, C_5) - \operatorname{lk}(C_2, C_3)\operatorname{lk}(C_1, C_4)\operatorname{lk}(C_5, C_1). \end{aligned}$$

Theorem 7 (Circles Meeting 4 Curves) *For any disjoint oriented smooth closed curves C_1, C_2, C_3, C_4 in $\mathbb{R}^3$ in general position, sets*

$$\begin{aligned} \mathcal{S}_{121234} &= \mathcal{S}(C_1, C_2, C_1, C_1, C_3, C_4) \\ \mathcal{S}_{121324} &= \mathcal{S}(C_1, C_2, C_1, C_3, C_2, C_4) \\ \mathcal{S}_{123124} &= \mathcal{S}(C_1, C_2, C_3, C_1, C_2, C_4) \end{aligned}$$

are 0–manifolds, which admit orientations ω_1, ω_2 and ω_3, respectively, such that

$$\begin{aligned} &\operatorname{sign}(\mathcal{S}_{121234}, \omega_1) = \\ &\qquad (\operatorname{lk}(C_1, C_2))^2 \operatorname{lk}(C_3, C_4) - \operatorname{lk}(C_2, C_1) \operatorname{lk}(C_2, C_3) \operatorname{lk}(C_4, C_1) \\ &\operatorname{sign}(\mathcal{S}_{121324}, \omega_2) = \\ &\qquad \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_1, C_3) \operatorname{lk}(C_2, C_4) - \operatorname{lk}(C_2, C_1) \operatorname{lk}(C_3, C_2) \operatorname{lk}(C_4, C_1) \\ &\operatorname{sign}(\mathcal{S}_{123124}, \omega_3) = \\ &\qquad \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_1) \operatorname{lk}(C_2, C_4) - \operatorname{lk}(C_2, C_3) \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_4, C_1). \end{aligned}$$

1.4 In High-Dimensional Spaces

The results presented above can be generalized to submanifolds of projective and Euclidean spaces of higher dimensions. Namely,

High-Dimensional Theorem 1 *For any disjoint oriented smooth closed submanifolds C_1, C_2, C_3, C_4 of $\mathbb{R}P^{2n+1}$ in general position with $\dim C_1 = \dim C_3 = p$, $\dim C_2 = \dim C_4 = q$ and $p+q = 2n$, the set $\mathcal{P} = \mathcal{P}(C_1, C_2, C_3, C_4)$ is a 0–manifold, which admits a natural orientation ω with signature*

$$\operatorname{sign}(\mathcal{P}, \omega) = 2\left(\operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_4) - \operatorname{lk}(C_2, C_3) \operatorname{lk}(C_4, C_1)\right).$$

Odd dimension of the projective space is necessary for its orientability and existence of linking numbers.

High-Dimensional Theorem 2 *For any disjoint oriented smooth closed submanifolds C_1, C_2, C_3, C_4 of $\mathbb{R}^n$ in general position with $\dim C_1 + \dim C_2 = n-1$ and $\dim C_3 + \dim C_4 = n-1$, the set $\mathcal{A} = \mathcal{A}(C_1, C_2, C_3, C_4)$ of lines meeting curves C_1, C_2, C_3, C_4 in their order is a 0–manifold, which admits a natural orientation ω with signature*

$$\operatorname{sign}(\mathcal{A}, \omega) = \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_4).$$

High-Dimensional Theorem 3 *For any disjoint oriented smooth closed submanifolds C_1 and C_2 of $\mathbb{R}^n$ in general position with $\dim C_1 + \dim C_2 = n-1$, the set $\mathcal{A} = \mathcal{A}(C_1, C_2, C_1, C_2)$ is a 0–dimensional manifold, which admits an orientation ω with signature*

$$\operatorname{sign}(\mathcal{A}, \omega) = (\operatorname{lk}(C_1, C_2))^2.$$

High-Dimensional Theorem 4 *For any disjoint oriented smooth closed submanifolds C_1, C_2 and C_3 of $\mathbb{R}^n$ in general position with $\dim C_2 = \dim C_3 = n - 1 - \dim C_1$, the sets $\mathcal{A}_{1213} = \mathcal{A}(C_1, C_2, C_1, C_3)$ and $\mathcal{A}_{1231} = \mathcal{A}(C_1, C_2, C_3, C_1)$ are 0–manifolds, which admit orientations ω_1 and ω_2, respectively, such that*

$$\operatorname{sign}(\mathcal{A}_{1213}, \omega_1) = \operatorname{sign}(\mathcal{A}_{1231}, \omega_2) = \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_1, C_3).$$

High-Dimensional Theorem 5 *For any disjoint oriented smooth closed submanifolds C_1, C_2, C_3, C_4, C_5, C_6 of $\mathbb{R}^n$ in general position with $\dim C_1 = \dim C_3 = \dim C_5 = p$, $\dim C_2 = \dim C_4 = \dim C_6 = q$ and $p + q = n - 1$, the set $\mathcal{S} = \mathcal{S}(C_1, C_2, C_3, C_4, C_5, C_6)$ is a 0–manifold, which admits an orientation ω with signature*

$$\operatorname{sign}(\mathcal{S}, \omega) = \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_4) \operatorname{lk}(C_5, C_6) - \operatorname{lk}(C_2, C_3) \operatorname{lk}(C_4, C_5) \operatorname{lk}(C_6, C_1).$$

High-Dimensional Theorem 6 *For any disjoint oriented smooth closed submanifolds C_1, C_2, C_3, C_4, C_5 of $\mathbb{R}^n$ in general position with $\dim C_1 = \dim C_4 = p$, $\dim C_2 = \dim C_3 = \dim C_5 = q$ and $p + q = n - 1$, the set*

$$\mathcal{S}_{121345} = \mathcal{S}(C_1, C_2, C_1, C_3, C_4, C_5)$$

is a 0–manifold with natural orientation ω such that

$$\operatorname{sign}(\mathcal{S}_{121345}, \omega) = \\ \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_1, C_3) \operatorname{lk}(C_4, C_5) - \operatorname{lk}(C_2, C_1) \operatorname{lk}(C_3, C_4) \operatorname{lk}(C_5, C_1).$$

If, moreover, $p = q$, the set

$$\mathcal{S}_{123145} = \mathcal{S}(C_1, C_2, C_3, C_1, C_4, C_5)$$

is a 0–manifold, which has a natural orientation ω_1 such that

$$\operatorname{sign}(\mathcal{S}_{123145}, \omega_1) = \\ \operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_1) \operatorname{lk}(C_4, C_5) - \operatorname{lk}(C_2, C_3) \operatorname{lk}(C_1, C_4) \operatorname{lk}(C_5, C_1).$$

High-Dimensional Theorem 7 *For any disjoint oriented smooth closed submanifolds C_1, C_2, C_3, C_4 in $\mathbb{R}^n$ in general position with $\dim C_1 = \dim C_3 = p$, $\dim C_2 = \dim C_4 = q$ and $p + q = n - 1$, the set*

$$\mathcal{S}_{121234} = \mathcal{S}(C_1, C_2, C_1, C_2, C_3, C_4)$$

is a 0–manifold with natural orientation ω_1 such that

$$\operatorname{sign}(\mathcal{S}_{121234},\omega) = (\operatorname{lk}(C_1,C_2))^2 \operatorname{lk}(C_3,C_4) - \operatorname{lk}(C_2,C_1)\operatorname{lk}(C_2,C_3)\operatorname{lk}(C_4,C_1).$$

If, moreover, $p = q$, the sets

$$\mathcal{S}_{121324} = \mathcal{S}(C_1,C_2,C_1,C_3,C_2,C_4)$$
$$\mathcal{S}_{123124} = \mathcal{S}(C_1,C_2,C_3,C_1,C_2,C_4)$$

are 0–manifolds, which admit orientations ω_1 and ω_2, respectively, such that

$$\operatorname{sign}(\mathcal{S}_{121324},\omega_1) = \operatorname{lk}(C_1,C_2)\operatorname{lk}(C_1,C_3)\operatorname{lk}(C_2,C_4) - \operatorname{lk}(C_2,C_1)\operatorname{lk}(C_3,C_2)\operatorname{lk}(C_4,C_1)$$
$$\operatorname{sign}(\mathcal{S}_{123124},\omega_2) = \operatorname{lk}(C_1,C_2)\operatorname{lk}(C_3,C_1)\operatorname{lk}(C_2,C_4) - \operatorname{lk}(C_2,C_3)\operatorname{lk}(C_1,C_2)\operatorname{lk}(C_4,C_1).$$

1.5 Quadrisecants in Literature

I am not aware about any publication, where the same problems were considered. However, there are several papers devoted to similar problems about quadrisecants of knots and links in $\mathbb{R}^3$. A *quadrisecant* of a link $L \subset \mathbb{R}^3$ is a line meeting L at least at four points. In other words, a quadrisecant of a link L is a point of $\mathcal{A}(L,L,L,L)$. In a sense, the problems about quadrisecants of a single curve are more sophisticated than the problems considered above. In particular, the problems about quadrisecants of a single curve do belong to low dimensional topology.

The most classical of the papers on quadrisecants is a dissertation of Erika Pannwitz [8] published in 1933. Here are the main two results of that paper:

Pannwitz Theorem 1. *Any generic piecewise linear knot in $\mathbb{R}^3$ which is not isotopic to the unknot has a quadrisecant (i.e., a line meeting the knot at four points).*

Pannwitz Theorem 2. *Any generic piecewise linear link of two components $L_1 \cup L_2 \subset \mathbb{R}^3$ which are linked in the sense that each represents a non-trivial homotopy class in the complement of the other has an essential quadrisecant, that is a line which intersects twice each of the components in such a way that the segment enclosed between the intersection points of the line with one component contains an intersection point with the other one.*

Besides, Pannwitz [8] proved low bounds for the number of quadrisecants in both cases.

In 1982 H.R.Morton and D.M.Q.Mond [7] proved the same results in the differential category, that is for generic smooth knots and links (their statement about two-component links was weaker: they assumed that the components have nonvanishing linking number). This part of their results follows also from Theorem 3.

In 1994 Greg Kuperberg [6] proved the following theorem extening, in particular, these results to arbitrary tame knot and links.

Kuperberg Theorem. *Every non-trivial tame link in R^3 has a quadrisecant.*

A recent paper [1] by Ryan Budney, James Conant, Kevin P. Scannell, and Dev Sinha contains a result on quadrisecants of a generic knot that admits a formulation very close in its spirit to the results of this paper. Quadrisecants of a generic knot K constitute a finite set. Some part of it can be equipped with a natural orientation such that its signature is the simplest non-trivaial Vassiliev-Goussarov invariant of K. This invariant is of degree two and called the Conway knot invariant.

1.6 Possible extension of classical results to projective links

Although Pannwitz-Morton-Mond Theorems formulated above make sense for links in the projective space $\mathbb{R}P^3$, both statements, and proofs require essential corrections.

In the proofs, the notion of convex hull is used at a crucial point. This notion is not applicable to a link in $\mathbb{R}P^3$.

As for the statements, in the projective space there are two isotopy types of knots which are obvious counter-parts for the unknot in $\mathbb{R}^3$: the types represented by a non-singular non-empty conic (e.g., circle) and projective line. Neither circle, nor slightly perturbed projective line (say, one component non-singular plane cubic curve) has a quadrisecant. However there exist knots which do not belong to these isotopy classes and nonetheless have no quadrisecant. For example, a knot 2_1 from my table [3] of prime projective knots with at most 6 crossings is isotopic to a real algebraic curve on a hyperboloid of bidegree $(3,1)$. In Figure 5 this curve is shown, together with the hyperboloid.

A line which does not lie on the hyperboloid meets the hyperboloid (and hence the curve) in at most two points, while a line on the hyperboloid meets the curve at one or three point (depending on which family of generatrices it belongs to). Thus the curve has no quadrisecant. Of course, the mirror image of this curve

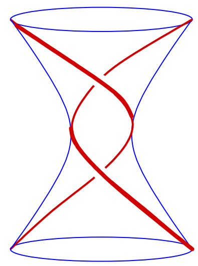

Figure 5:

has no quadrisecant, too. An easy modification of arguments used by Pannwitz and Kuperberg, shows that:

Any tame knot in $\mathbb{R}P^3$ *without quadrisecants belongs to one of the four isotopy types listed above.*

Even more changes are required in the case of links. There exist infinitely many isotopy types of two-component links in the projective space which have no essential quadrisecant. Indeed, this property has any link with components separated by a hyperboloid.

1.7 Gratitudes

I am grateful to Oleg Viro for interest to my work and useful discussions. I am also grateful to Mathematics Department of Uppsala University and MSRI for partial support of this work.

2 Proofs

Proofs of all theorems formulated above are based on the same ideas. In the simplest and most profound way these ideas work in the proof of Theorem 1. Therefore, we concentrate first on this theorem.

2.1 Surfaces of Secants

Let C_1, C_2 and C_3 be three smooth disjoint closed curves in $\mathbb{R}P^3$. Similarly to the notation introduced above, denote by $\mathcal{P}(C_1, C_2, C_3)$ the set of projective lines each of which passes through points $a \in C_1$, $b \in C_2$, $c \in C_3$.

Projective lines belonging to $\mathcal{P}(C_1, C_2, C_3)$ are called *common secants* of C_1, C_2 and C_3. The union of all common secants of C_1, C_2 and C_3 is called the *secant surface* of C_1, C_2 and C_3 and denoted by $s(C_1, C_2, C_3)$. Thus

$$s(C_1, C_2, C_3) = \bigcup_{L \in \mathcal{P}(C_1, C_2, C_3)} L.$$

The role of $s(C_1, C_2, C_3)$ becomes clear from the following simple remark: given any curve $C_4 \subset \mathbb{R}P^3$, for each point $x \in s(C_1, C_2, C_3) \cap C_4$ there exists a line passing through x and meeting C_1, C_2, C_3, C_4. The cyclic order, in which this line meets C_1, C_2, C_3, C_4, depends on position of x on $s(C_1, C_2, C_3)$ with respect to curves C_1, C_2 and C_3. See Figure 6.

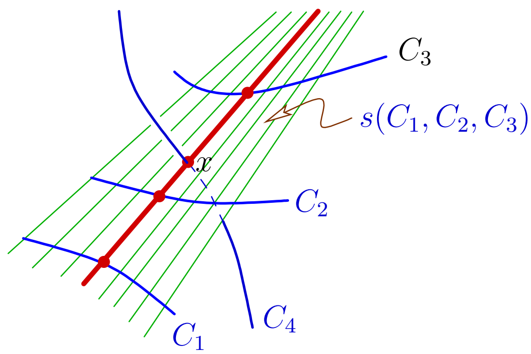

Figure 6:

Resolution of singularities. For most triples of curves $s(C_1, C_2, C_3)$ is not a 2-manifold. If the curves C_i are algebraic, $s(C_1, C_2, C_3)$ is an algebraic surface, but usually with lots of singular points. In what follows, we use a standard trick to resolve most of the singularities. For generic curves, the trick resolves all singularities.

A common secant L of curves C_1, C_2, and C_3 with three marked points p_i, $i = 1, 2, 3$ chosen from $C_i \cap L$ is called a *pointed secant.* Denote by $S(C_1, C_2, C_3)$ the set

$$\{(x, p_1, p_2, p_3) \in \mathbb{R}P^3 \times C_1 \times C_3 \times C_3 \mid \text{ points } x, p_1, p_2, p_3 \text{ are colinear}\}$$

One may think on $S(C_1, C_2, C_3)$ as the union of all pointed secants of C_1, C_2, and C_3. Let us call $S(C_1, C_2, C_3)$ the *pointed secant surface* of C_1, C_2, C_3. Denote the *set* of all pointed secants by $T(C_1, C_2, C_3)$. Since a pointed secant is defined by the three marked points, we can identify $T(C_1, C_2, C_3)$ with

$$\{(p_1, p_2, p_3) \in C_1 \times C_2 \times C_3 \mid \text{ points } p_1, p_2, p_3 \text{ are colinear}\}$$

Notice natural inclusions

$$s(C_1, C_2, C_3) \subset \mathbb{R}P^3,$$
$$S(C_1, C_2, C_3) \subset \left(\mathbb{R}P^3\right)^4,$$
$$T(C_1, C_2, C_3) \subset \left(\mathbb{R}P^3\right)^3.$$

Using these inclusions, induce topology on $s(C_1, C_2, C_3)$, $S(C_1, C_2, C_3)$ and $T(C_1, C_2, C_3)$. The natural map

$$u : S(C_1, C_2, C_3) \to s(C_1, C_2, C_3) : (x, p_1, p_2, p_3) \mapsto x$$

forgetting the marked points is continuous with respect to these natural topological structures. Denote the composition of this map with the inclusion $s(C_1, C_2, C_3) \hookrightarrow \mathbb{R}P^3$ by U. Thus, U is a natural map $S(C_1, C_2, C_3) \to \mathbb{R}P^3$ acting by formula $(x, p_1, p_2, p_3) \mapsto x$ and having image $s(C_1, C_2, C_3)$.

The natural map

$$q : S(C_1, C_2, C_3) \to T(C_1, C_2, C_3) : (x, p_1, p_2, p_3) \mapsto (p_1, p_2, p_3)$$

is a locally trivial fibration with fiber circle. The points marked on secants provide three disjoint sections of this fibration defined by formulas $(p_1, p_2, p_3) \mapsto (p_i, p_1, p_2, p_3)$ with $i = 1, 2, 3$. The images of these sections can be described by the following formulas

$$P_i = \{(x, p_1, p_2, p_3) \in S(C_1, C_2, C_3) \mid x = p_i\}.$$

A fiber $q^{-1}(p_1, p_2, p_3)$ of q is naturally identified with the projective line in $\mathbb{R}P^3$ passing through (colinear) points p_1, p_2 and p_3. There exists a unique projective transformation $q^{-1}(p_1, p_2, p_3) \to \mathbb{R}P^1$ sending $p_1 \mapsto (1:0)$, $p_2 \mapsto (1:1)$ and $p_3 \mapsto (0:1)$. These transformations altogether define a homeomorphism

$$S(C_1, C_2, C_3) \to \mathbb{R}P^1 \times T(C_1, C_2, C_3),$$

under which P_1, P_2 and P_3 are mapped to fibers $(1:0) \times T(C_1, C_2, C_3)$, $(1:1) \times T(C_1, C_2, C_3)$, and $(0:1) \times T(C_1, C_2, C_3)$, respectively.

2.2 Genericity assumptions

For a pointed secant (p_1, p_2, p_3) of C_1, C_2, C_3, denote the tangent line of C_i at p_i by L_i. A pointed secant (p_1, p_2, p_3) is said to be *regular,* if the lines L_i with $i = 1, 2, 3$ are pairwise disjoint.

Along a regular pointed secant, a germ of $U(s(C_1, C_2, C_3))$ is C^1-approximated by a germ of surface $s(L_1, L_2, L_3)$, which is a hyperboloid. See Figure 7.

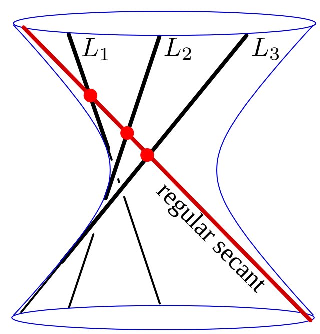

Figure 7:

If pointed secant (p_1, p_2, p_3) is not regular, that is, say L_i, L_j are coplanar for some $i, j \in \{1, 2, 3\}$ with $i \neq j$, then the projection of the curves C_i and C_j from the third of the marked points, say p_k, are tangent to each other at the images of p_i and p_j. If this tangency is quadratic and L_k is coplanar neither to L_i, nor to L_j, then the pointed secant line (p_1, p_2, p_3) is said to be *almost regular* and p_k is called its *special point.* See Figure 8.

The surface $s(L_1, L_2, L_3)$ is a union of two planes. It C^1-approximates the germ of U along the pointed secant at all points of the secant except p_k. To get a sufficient jet at p_k, one can replace C_k still with line L_k tangent to C_k at p_k, while the other two curves should be replaced by appropriate conics.

A good model for a germ of $U(s(C_1, C_2, C_3))$ near an almost regular pointed secant is Whitney umbrella. In fact Whitney umbrella is $s(C_1, C_2, C_3)$, where C_1 is a line, C_2 and C_3 are parabolas with axes parallel to C_1 and symmetric to each other against C_1. See Figure 9.

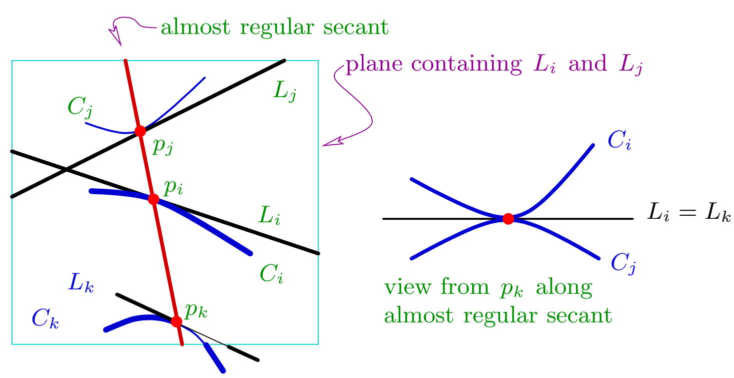

Figure 8:

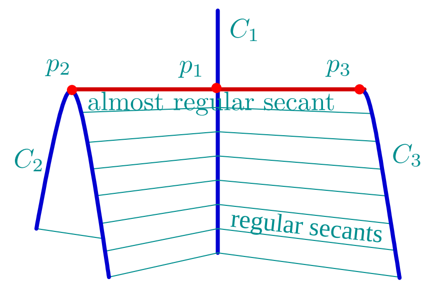

Figure 9:

For arbitrary C_1, C_2 and C_3, along an almost regular pointed secant a germ of $U(s(C_1, C_2, C_3))$ is approximated by a germ of a surface projectively equivalent to the Whitney umbrella.

A triple C_1, C_2, C_3 of disjoint smooth closed curves in the projective space is called *regular,* if all but finitely many of its pointed secants are regular and each non-regular pointed secant is almost regular.

Lemma 1 *Regularity is a generic property of a triple of disjoint smooth closed curves.*

For the notion of generic property, we refer to a general theory of general position presented by Wall [9]. Cf. [6]. The proof is straightforward. In fact, Lemma 1 is almost entirely covered by Lemma 2.4 from the paper [6] of Kuperberg. □

To understand regularity and almost regularity of secants, take a pointed secant (p_1, p_2, p_3) and consider a view on C_2 and C_3 from viewpoint p_1.

If this is a regular secant, along it we will see crossing point of C_2 and C_3. If the same line is not involved in other pointed secants, the crossing is simple transversal. It is stable: under a small move of p_1 along C_1 the secant moves, and the other two points p_2 and p_3 move along C_2 and C_3 with speeds determined by the speed of p_1.

If this is an almost regular secant and p_1 is its special point, C_2 and C_3 will look tangent to each other quadratically. This tangency is not stable: under a small move of p_1 in one direction along C_1 the point of tangency splits into two crossings, under a move in the opposite direction it disappears. Passing p_1 along C_1 gives rise to a second Reidemeister move.

Secants versus Reidemeister moves. This interpretation of regularity and almost regularity relates Lemma 1 to facts well-known from the knot theory.

Indeed, it is well-known that a projection of two smooth curves from a generic point has only transversal double points, and in a one parameter generic family only Reidemeister moves happen at isolated moments. Thus when we travel along one of three generic curves under consideration, we meet only regular secants at all but finite number of points. Since we are interested only in intersection points of images of two different branches, the first Reidemeister move of the picture seen from our moving viewpoint is not of any value. The third Reidemeister moves correspond to the lines which are underlying for more than one pointed secant, each of which are regular. Second Reidemeister moves correspond to almost regular secants.

Lemma 2 (**Pointing resolves singularities**) *For any regular triple C_1, C_2, C_3 of smooth closed curves in $\mathbb{R}P^3$, the set $T(C_1, C_2, C_3)$ of pointed secants is a smooth 1-dimensional submanifold of $\left(\mathbb{R}P^3\right)^3$, the pointed secant surface $S(C_1, C_2, C_3)$ is a smooth two-dimensional submanifold of $\left(\mathbb{R}P^3\right)^4$ and $U : S(C_1, C_2, C_3) \to \mathbb{R}P^3$ is a differentiable map. The only singularities of U are pinch points at the special points of non-regular pointed secants.*

We skip the proof. It is a straightforward application of Implicit Function Theorem and Whitney's characterization of a pinch point. □

Lemma 3 (**Topology of pointed secant surface**) *For any regular triple C_1, C_2, C_3 of smooth closed curves in $\mathbb{R}P^3$, each connected component of pointed secant surface $S(C_1, C_2, C_3)$ is diffeomorphic to torus $S^1 \times S^1$ under diffeomorphism which maps each pointed secant line to a fiber $pt \times S^1$, while the sections P_1, P_2, P_3 of the fibration $S(C_1, C_2, C_3) \to T(C_1, C_2, C_3)$ corresponding to the marked points are mapped to three fibers of the complementary family, $S^1 \times pt$.*

This is an immediate corollary of Lemma 2. □

2.3 Orientation of pointed secant surface

Along a regular secant. Let C_1, C_2 and C_3 be oriented smooth closed curves in $\mathbb{R}P^3$ forming a regular triple. Denote by O_i the given orientation of C_i. (Orientation is considered as a function taking values ± 1 on bases of each tangent space.)

Let (p_1, p_2, p_3) be a regular pointed secant line of C_1, C_2 and C_3. Denote by L the line in $\mathbb{R}P^3$ passing through p_1, p_2 and p_3, and by L_i the tangent line of

C_i at p_i. Orient L_i according to the orientation of C_i. Recall that regularity of the pointed secant means that these lines are pairwise disjoint. Therefore, each pair L_i, L_j of them with $i \neq j$ has a well-defined linking number $\operatorname{lk}(L_i, L_j)$ equal to $\pm\frac{1}{2}$.

Consider hyperboloid $s(L_1, L_2, L_3)$, which osculates secant surface

$$s(C_1, C_2, C_3) = U(S(C_1, C_2, C_3))$$

along L. For any tangent vector e of $S(C_1, C_2, C_3)$, its image $dU(e)$ is a tangent vector of $s(L_1, L_2, L_3)$. The line L (as a line on a hyperboloid) is a circle embedded two-sidedly in $s(L_1, L_2, L_3)$. Choose one of the sides and choose a non-zero vector e_i at p_i tangent to C_i and directed towards the chosen side of L on $s(L_1, L_2, L_3)$.

Lemma 4 *Let σ be a permutation of $(1, 2, 3)$. Then*

$$\operatorname{lk}(L_{\sigma(2)}, L_{\sigma(3)}) O_{\sigma(1)}(e_{\sigma(1)})$$

does not depend on σ.

Proof Denote line L_i oriented along e_i by $\overline{L_i}$. Recall (see [5]) that the product $\operatorname{lk}(L_1, L_2) \operatorname{lk}(L_2, L_3) \operatorname{lk}(L_3, L_1)$ of all pairwise linking numbers of three pairwise disjoint oriented lines in $\mathbb{R}P^3$ does not depend on the orientations of the lines. In particular,

$$\operatorname{lk}(L_1, L_2) \operatorname{lk}(L_2, L_3) \operatorname{lk}(L_3, L_1) = \operatorname{lk}(\overline{L_1}, \overline{L_2}) \operatorname{lk}(\overline{L_2}, \overline{L_3}) \operatorname{lk}(\overline{L_3}, \overline{L_1}).$$

All factors in the right hand side of this equality are equal, because oriented links $\overline{L_i} \cup \overline{L_j}$ are isotopic to each other. Hence

$$4 \operatorname{lk}(\overline{L_1}, \overline{L_2}) \operatorname{lk}(\overline{L_2}, \overline{L_3}) \operatorname{lk}(\overline{L_3}, \overline{L_1}) = \operatorname{lk}(\overline{L_{\sigma(2)}}, \overline{L_{\sigma(3)}}).$$

Change of the orientation of a line multiplies the linking number of this line with other line by -1. Hence

$$\operatorname{lk}(\overline{L_{\sigma(2)}}, \overline{L_{\sigma(3)}}) = \operatorname{lk}(L_{\sigma(2)}, L_{\sigma(3)}) O_{\sigma(2)}(e_{\sigma(2)}) O_{\sigma(3)}(e_{\sigma(3)}).$$

Summarizing we get

$$4 \operatorname{lk}(L_1, L_2) \operatorname{lk}(L_2, L_3) \operatorname{lk}(L_3, L_1) = \operatorname{lk}(L_{\sigma(2)}, L_{\sigma(3)}) O_{\sigma(2)}(e_{\sigma(2)}) O_{\sigma(3)}(e_{\sigma(3)}).$$

Multiplying this equality by $O_1(e_1) O_2(e_2) O_3(e_3)$ we obtain an expression for $\operatorname{lk}(L_{\sigma(2)}, L_{\sigma(3)}) O_{\sigma(1)}(e_{\sigma(1)})$ independent of σ:

$$\operatorname{lk}(L_{\sigma(2)}, L_{\sigma(3)}) O_{\sigma(1)}(e_{\sigma(1)}) = \\ 4 \operatorname{lk}(L_1, L_2) \operatorname{lk}(L_2, L_3) \operatorname{lk}(L_3, L_1) O_1(e_1) O_2(e_2) O_3(e_3).$$

$\square$

The given ordering of curves C_i, $i = 1, 2, 3$ defines a cyclic order of points p_i, $i = 1, 2, 3$, which, in turn, defines an orientation of L. For each $i = 1, 2, 3$, choose a non-zero vector f_i at p_i tangent to L and directed along the orientation of L.

Let V be a connected neighborhood of L in $S(C_1, C_2, C_3)$ disjoint from non-regular pointed secants. Without loss of generality we can assume that V is a union of pointed secants. Since V does not meet non-regular pointed secants, the restriction of U to V is an immersion. Moreover, since the restriction of U to L is embedding, we can choose V small enough to make $U|_V$ an embedding.

Vectors e_i, f_i form a basis of the tangent space of V at p_i. The bases (e_1, f_1), (e_2, f_2) and (e_3, f_3) define the same orientation of V. Let us correct it by $\frac{1}{2}\operatorname{lk}(L_2, L_3)O_1(e_1)$, that is consider the orientation O of V which takes value $\frac{1}{2}\operatorname{lk}(L_2, L_3)O_1(e_1)$ on (e_1, f_1). By Lemma 4, O takes values $\frac{1}{2}\operatorname{lk}(L_3, L_1)O_2(e_2)$ on (e_2, f_2) and $\frac{1}{2}\operatorname{lk}(L_1, L_2)O_3(e_3)$ on (e_3, f_3).

Extending the orientation across almost regular secants. This orientation can be defined at any point of a regular pointed secant, the construction depends continuously of the point. Thus, we have defined an orientation of the complement of non-regular pointed secants in $S(C_1, C_2, C_3)$. This orientation depends on the orientation of each curve C_i and a cyclic order of these curves.

Lemma 5 *Orientation O extends across almost regular pointed secants to the whole $S(C_1, C_2, C_3)$.*

Proof When (p_1, p_2, p_3) moves in $T(C_1, C_2, C_3)$ and passes through an almost regular secant $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ with special point $\bar{p}_1$, both the linking number $\operatorname{lk}(L_2, L_3)$, and $O_1(e_1)$ change and, hence, their product does not change.

Indeed, by the definition of almost regular pointed secant, the projections of C_2 and C_3 from $\bar{p}_1$ are tangent to each other at the images of $\bar{p}_2$ and $\bar{p}_3$. The tangency is quadratic, so, when the center of the projection p_1 moves along C_1 passing $\bar{p}_1$, the projection of $C_2 \cup C_3$ experiences the second Reidemeister move. When p_1 is on one side of $\bar{p}_1$, there are two intersection points of the images of C_2 and C_3 which are close to the image of non-regular secant under projection from $\bar{p}_1$, when p_1 gets to the other side, the intersection points disappear. The vanishing pair of intersection points corresponds to a pair of pointed secant lines, say (p_1, p_2^+, p_3^+) and (p_1, p_2^-, p_3^-) close to $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$. On curve $T(C_1, C_2, C_3)$ they are on the opposite sides of $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$. Thus, $\bar{p}_1$ is the turning point for the map $U|_{P_1}$, and $O_1(e_1)$ changes the sign, when one jumps from (p_1, p_2^+, p_3^+) to (p_1, p_2^-, p_3^-).

At the intersection points of the projections of C_2 and C_3 from p_1 corresponding to these lines, the writhe numbers are opposite to each other. These writhe number equals the doubled linking number of the lines tangent to the branches. Therefore the linking number changes when (p_1, p_2, p_3) passes $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$. $\square$

2.4 Orientation of the pointed sections

As above, let C_1, C_2 and C_3 be oriented smooth closed curves in $\mathbb{R}P^3$ forming a regular triple. Each fiber of the fibration $S(C_1, C_2, C_3) \to T(C_1, C_2, C_3)$ intersects sections P_1, P_2 and P_3 in three points, p_1, p_2, p_3, respectively, and is divided by these intersection points into three arcs. Denote by $\Sigma_{i,j}$ the union of closures of those arcs which connect p_i with p_j. This is a compact surface with boundary $P_i \cup P_j$.

Equip $\Sigma_{i,j}$ with the orientation induced by O. Equip P_1, P_2 and P_3 with the orientations such that $\partial\Sigma_{1,2} = P_1 \cup (-P_2)$, $\partial\Sigma_{2,3} = P_2 \cup (-P_3)$ and $\partial\Sigma_{3,1} = P_3 \cup (-P_1)$.

2.5 Degree of map of a pointed section to curve

Lemma 6 *The degree of map $U_i : P_i \to C_i$ defined by the natural map $U : S(C_1, C_2, C_3) \to \mathbb{R}P^3$ is equal to $2\,\mathrm{lk}(C_j, C_k)$ with $\{i, j, k\} = \{1, 2, 3\}$.*

Proof Consider the case $i = 1$. Choose a point $p_1 \in C_1$ such that each common secant of C_1, C_2, C_3 passing through p_1 is regular. This means that the projection of $C_2 \cup C_3$ from p_1 is generic and p_1 is a regular value of $U_1 : P_1 \to C_1$. The preimage of p_1 under U_1 consists of points (p_1, p_1, p_2^k, p_3^k) of pointed secant lines (p_1, p_2^k, p_3^k) passing through p_1. So, we have to prove that the sum of local degrees of U_1 over all these points is $2\,\mathrm{lk}(C_2, C_3)$.

At point (p_1, p_1, p_2^k, p_3^k) choose the vector e_1 such that basis (e_1, f_1) defines the orientation O. Since f_1 is outwards normal vector for $\Sigma_{3,1}$ at (p_1, p_1, p_2^k, p_3^k), vector e_1 defines the orientation of P_1 at this point. Hence the local degree of U_1 at this point is $O_1(e_1)$.

Due to our choice of e_1, the value of O on (e_1, f_1) is $+1$. On the other hand, this value, by the definition of O, is $2\,\mathrm{lk}(L_2, L_3) O_1(e_1)$. Therefore

$$O_1(e_1) = 2\,\mathrm{lk}(L_2, L_3).$$

The left hand side is the local degree of U_1 at (p_1, p_1, p_2^k, p_3^k). The right hand side is the local writhe of the projection of $C_2 \cup C_3$ from p_1 at the image of p_2^k.

The sum of local writhe numbers at all the intersection points of the projections of C_2 and C_3 is equal to $2\,\mathrm{lk}(C_2, C_3)$, see, e.g. [4]. $\square$

Lemma 7 *The image of a fundamental cycle of $\Sigma_{3,1}$ under U is a smooth singular chain with boundary $2\,(\mathrm{lk}(C_1, C_2)[C_3] - \mathrm{lk}(C_2, C_3)[C_1])$.*

This is an immediate corollary of Lemma 6. $\square$

Lemma 8 *The intersection number of $U_*[\Sigma_{3,1}]$ with an oriented closed curve C_4 disjoint from $C_1 \cup C_2 \cup C_3$ is*

$$2\det\begin{pmatrix} \mathrm{lk}(C_1, C_2) & \mathrm{lk}(C_1, C_4) \\ \mathrm{lk}(C_3, C_2) & \mathrm{lk}(C_3, C_4) \end{pmatrix}$$

2.6 Proof of Theorem 1

Let C_1, C_2, C_3 and C_4 be disjoint oriented smooth closed curves in $\mathbb{R}P^3$ generic in the sense of Theorem 1 (see Section 1). To prove Theorem 1, we have to find an orientation ω of 0-manifold $\mathcal{P}(C_1, C_2, C_3, C_4)$ such that

$$\mathrm{sign}(\mathcal{P}(C_1, C_2, C_3, C_4), \omega) = 2\det\begin{pmatrix} \mathrm{lk}(C_1, C_2) & \mathrm{lk}(C_1, C_4) \\ \mathrm{lk}(C_3, C_2) & \mathrm{lk}(C_3, C_4) \end{pmatrix}.$$

The genericity condition implies that any $L \in \mathcal{P}(C_1, C_2, C_3, C_4)$ can be turned into a pointed common secant of C_1, C_2 and C_3 in a unique way, and the pointed common secant is regular.

Therefore by a small perturbation of C_1, C_2 and C_3 which is trivial on a neighborhood of each common secant of all four curves, we can make the triple C_1, C_2, C_3 regular without changing $\mathcal{P}(C_1, C_2, C_3, C_4)$. Thus we can assume that triple of curves C_1, C_2, C_3 is regular from the very beginning.

There is obvious bijection between $\mathcal{P}(C_1, C_2, C_3, C_4)$ and $C_4 \cap U(\Sigma_{3,1})$. The intersection of C_4 with $U(\Sigma_{3,1})$ is transversal by the fourth condition of general position, see Section 1.1. The orientations of C_4, $\Sigma_{1,3}$ and the ambient space $\mathbb{R}P^3$ define an orientation of the intersection $C_4 \cap U(\Sigma_{3,1})$ turning it into an oriented 0-manifold. The intersection number $U(\Sigma_{1,3}) \circ C_4$ is the signature of $C_4 \cap U(\Sigma_{3,1})$. $\square$

2.7 In Affine Space

Let C_1, C_2, C_3 be three smooth disjoint curves in $\mathbb{R}^3$. Denote by $\mathcal{A}(C_1, C_2, C_3)$ the set of affine lines in $\mathbb{R}^3$ each of which passes through points $a \in C_1$, $b \in C_2$ and $c \in C_3$ in this order. Put $sa(C_1, C_2, C_3) = \bigcup_{L \in \mathcal{A}(C, C_2, C_3)} L$. Denote by $SA(C_1, C_2, C_3)$ the set

$$\{(x, p_1, p_2, p_3) \in \mathbb{R}^3 \times C_1 \times C_3 \times C_3 \mid x, p_1, p_2, p_3 \text{ are colinear, } p_2 \in [p_1, p_3]\}$$

and by $TA(C_1, C_2, C_3)$ the set

$$\{(p_1, p_2, p_3) \in C_1 \times C_2 \times C_3 \mid p_1, p_2, p_3 \text{ are colinear, } p_2 \in [p_1, p_3]\}.$$

There are natural maps

$$\begin{aligned} ua &: SA(C_1, C_2, C_3) \to sa(C_1, C_2, C_3) : (x, p_1, p_2, p_3) \mapsto x \\ UA &: SA(C_1, C_2, C_3) \to \mathbb{R}^3 : (x, p_1, p_2, p_3) \mapsto x \\ qa &: SA(C_1, C_2, C_3) \to TA(C_1, C_2, C_3) : (x, p_1, p_2, p_3) \mapsto (p_1, p_2, p_3) \end{aligned}$$

Map qa is a trivial line fibration with trivialization

$$SA(C_1, C_2, C_3) \to \mathbb{R}^1 \times T(C_1, C_2, C_3),$$

under which the sections P_1, P_3 defined by

$$P_i = \{(x, p_1, p_2, p_3) \in S(C_1, C_2, C_3) \mid x = p_i\}.$$

are mapped to fibers $0 \times T(C_1, C_2, C_3)$ and $1 \times T(C_1, C_2, C_3)$, respectively.

Inclusion $in : \mathbb{R}^3 \subset \mathbb{R}P^3$ induces embeddings

$$\begin{aligned} sa(C_1, C_2, C_3) &\to s(in(C_1), in(C_2), in(C_3)) \\ SA(C_1, C_2, C_3) &\to S(in(C_1), in(C_2), in(C_3)) \\ TA(C_1, C_2, C_3) &\to T(in(C_1), in(C_2), in(C_3)) \end{aligned}$$

However, they are not necessarily surjective, because in the definition of the spaces for affine situation there is condition $p_2 \in [p_1, p_3]$, which does not make sense and has no counter-part in the projective situation. Genericity assumptions considered in Section 2.2 above, can be borrowed entirely (although one could reduce them). The orientation of $S(C_1, C_2, C_3)$ introduced in Section 2.3 under the genericity assumptions can be repeated without changes in the affine situation, or can be borrowed using the embedding $SA(C_1, C_2, C_3) \to S(in(C_1), in(C_2), in(C_3))$.

Denote by Σ the union of rays $[p_3, \infty)$ on the fibers of fibration

$$qa : SA(C_1, C_2, C_3) \to TA(C_1, C_2, C_3).$$

Clearly, $\partial\Sigma = P_3$.

The following statement is a counter-part of Lemma 6.

Lemma 9 *The degree of map $UA_3 : P_3 \to C_3$ defined by the natural map $UA : SA(C_1, C_2, C_3) \to \mathbb{R}^3$ is equal to* $\operatorname{lk}(C_1, C_2)$.

Proof is similar to the proof of Lemma 6 given above. We consider the only point where the proofs differ.

The factor 2 disappeared, because the preimage of p_3 under UA_3 consists of points (p_1, p_2, p_3, p_3) of pointed secants (p_1, p_2, p_3) with $p_1 \in C_1$, $p_2 \in C_2$ and $p_2 \in [p_1, p_3]$. Thus we count, with appropriate signs, crossing points, where C_2 is above C_1, of the diagram for $C_1 \cup C_2$ generated by projection centered at p_3 . This gives the linking number. In the projective case (in Lemma 6) the condition $p_2 \in [p_1, p_3]$ was absent which resulted the factor 2. $\square$

Lemma 10 *The image of the fundamental cycle of Σ under mapping $UA : SA(C_1, C_2, C_3) \to \mathbb{R}^3$ is a smooth singular chain with closed (non-compact) support and boundary* $\operatorname{lk}(C_1, C_2)[C_3]$.

This is the counter-part of Lemma 7 and it follows immediately from Lemma 9. $\square$

Lemma 11 *The intersection number of $UA_*[\Sigma]$ with an oriented closed curve C_4 disjoint from $C_1 \cup C_2 \cup C_3$ is* $\operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_4)$.

$\square$

After this point the proof of Theorem 2 runs like its counter-part in Section 2.6. The 0-dimensional manifold $\mathcal{A}(C_1, C_2, C_3, C_4)$ is identified with $C_4 \cap UA(\Sigma)$.

Theorems 3 and 4 are proved in the exactly same way, although they cannot be deduced formally from Theorem 2.

2.8 Circles Meeting Curves

Lemma 12 *For any disjoint oriented smooth closed curves C_1, C_2, C_3 and C_4 in $\mathbb{R}^3$ and a point $p \in \mathbb{R}^3$ the set $\mathcal{S}(C_1, C_2, C_3, C_4, p)$ of circles meeting C_1, C_2, C_3, C_4 and p in this cyclic order is a 0-manifold, which admits orientation ω with signature* $\operatorname{lk}(C_1, C_2) \operatorname{lk}(C_3, C_4)$.

Proof Apply an inversion I of R^3 centered at p. This turns the circles belonging to $\mathcal{S}(C_1, C_2, C_3, C_4, p)$ to lines belonging to $\mathcal{A}(I(C_1), I(C_2), I(C_3), I(C_4))$. Now use Theorem 2. $\square$

For a generic collection C_1, C_2, C_3, C_4 and C_5 of smooth closed curves in $\mathbb{R}^3$ denote by $SC(C_1, C_2, C_3, C_4, C_5)$ the set of $(x, p_1, p_2, p_3, p_4, p_5) \in \mathbb{R}^3 \times C_1 \times C_2 \times C_3 \times C_4 \times C_5$ such that there exists a circle passing through p_1, p_2, p_3, p_4, p_5 in this cyclic order and x lies on this circle, too.

Denote by $TC(C_1, C_2, C_3, C_4, C_5)$ the set of $(p_1, p_2, p_3, p_4, p_5) \in C_1 \times C_2 \times C_3 \times C_4 \times C_5$ such that there exists a circle passing through p_1, p_2, p_3, p_4, p_5 in this cyclic order.

Formula $(x, p_1, p_2, p_3, p_4, p_5) \mapsto (p_1, p_2, p_3, p_4, p_5)$ defines a fibration

$$SC(C_1, C_2, C_3, C_4, C_5) \to TC(C_1, C_2, C_3, C_4, C_5)$$

with fiber circle. It has five disjoint sections defined by formulas

$$(p_1, p_2, p_3, p_4, p_5) \mapsto (p_i, p_1, p_2, p_3, p_4, p_5) \text{ with } i = 1, \dots, 5.$$

The images of these sections can be described by the following formulas:

$$P_i = \{(x, p_1, p_2, p_3, p_4, p_5) \in SC(C_1, C_2, C_3, C_4, C_5) \mid x = p_i\}.$$

The part of $SC(C_1, C_2, C_3, C_4, C_5)$ bounded by sections P_i and P_{i+1} and disjoint with other P_j is denoted by Σ_i (here, when we write $i+1$, we treat i modulo 5).

Similarly to what was done in Section 2.3, for generic collection of curves the space $SC(C_1, C_2, C_3, C_4, C_5)$ is equipped with a natural orientation. Equip sections P_i with orientations such that $\partial \Sigma_i = P_i \cup (-P_{i+1})$.

Lemma 13 *The degree of map $UC_i P_i \to C_i$ defined by the natural map*

$$UC : SC(C_1, C_2, C_3, C_4, C_5) \to \mathbb{R}^3 : (x, p_1, p_2, p_3, p_4, p_5) \mapsto x$$

is equal to $\operatorname{lk}(C_{i+1}, C_{i+2}) \operatorname{lk}(C_{i+3}, C_{i+4})$ *(here again the indices are treated as elements of $\mathbb{Z}/5\mathbb{Z}$).*

This lemma is similar to Lemmas 6 and 9. The proof of this lemma is based on Lemma 12.

After this point our proof of Theorem 5 goes along the same scheme as proofs of Theorems 1 and 2 above. We identify $\mathcal{S}(C_1, C_2, C_3, C_4, C_5, C_6$ with $C_6 \cap UC(\Sigma_5)$ and orient the latter using the given orientation of C_6 and the orientation of Σ_5.

Theorems 6 and 7 are proved by the same arguments, but applied to spaces appropriately changed.

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